

Unsharp eigenvalues and quantum contextuality

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The Kochen-Specker theorem, Bell inequalities, and several other tests that were designed to rule out hidden-variable theories, assume the existence of observables having infinitely sharp eigenvalues. A paradigmatic example is spin-1/2. It is measured with a Stern-Gerlach array whose outputs are divided into two classes, spin-up and spin-down, in correspondence to the two spots observed on a detection screen. The spot's finite size is attributed to imperfections of the measuring device. This assumption turns the experimental output into a dichotomic, discrete one, thereby allowing the assignment of each spot to an infinitely sharp eigenvalue. Alternatively, one can assume that the spot's finite size stems from eigenvalues spanning a continuous range. Can we disprove such an assumption? Can we rule out hidden-variable theories that reproduce quantum predictions by assuming that, e.g., the electron's magnetic moment is not exactly the same for all electrons? We address these questions by focusing on the Peres-Mermin version of the Bell-Kochen-Specker theorem. It is shown that the assumption of unsharp eigenvalues precludes ruling out non-contextual hidden-variable theories and hence quantum contextuality does not arise. Analogous results hold for Bell-like inequalities. This represents a new loophole that spoils several fundamental tests of quantum mechanics and issues the challenge to close it.

INTRODUCTORY REMARKS

There is a fundamental prescription in quantum mechanics (QM) that has been once qualified as a “precept of the founders” [1], namely the claim that it makes no sense to assign values to unmeasured observables. Such a precept was turned into a theorem by Bell [2] and independently by Kochen and Specker [3], being since referred to as the Bell-Kochen-Specker (BKS) theorem. It shows that it is impossible to construct a non-contextual hidden-variable (HV) theory that reproduces the predictions of QM. Non-contextuality means that the results obtained by measuring an observable are independent of any previous or simultaneous measurements on other, compatible [4] observables. The impossibility of constructing a non-contextual HV theory is often expressed by saying that QM is contextual. Now, such a feature of QM – though being perhaps somewhat peculiar – does not seem to frontally collide with common sense nor with possible approaches that might be undertaken when pursuing scientific endeavors. Indeed, contextual models in the social and in the natural sciences are perfectly acceptable and imply no conflict with common sense. But the claim that we cannot even assume that an observable has a value before it has been measured is certainly at odds with common sense. Such a feature sets QM apart from the rest of science. The idea that the value of an observable comes into being just through its measurement is something that conflicts with our most basic notions of reality; a reality that keeps existing – so we think – even if we do not interact with it. On the other hand, if we assume that precise values can be assigned to unmeasured observables, then we run into logical contradictions, as the BKS theorem shows. Thus, it seems that we must pay a high price in order to provide the quantum formalism with a self-consistent ontology. This price

is the abandonment of our most basic notions of reality, something we are reluctant to do even as practitioners of quantum physics. Classical ontology – according to which measurements of observables just reveal preexisting values – must be replaced by quantum ontology, if we want the quantum formalism to be not merely a computational tool, but a consistent model of the real world; a world in which we include ourselves, if necessary, as perceiving subjects. Now, in spite of all these needs we keep talking and thinking in terms of a classical ontology. This hints at a latent conflict between the quantum formalism and its interpretation in terms of our deeply rooted notions of reality. The BKS theorem brought this conflict into clearest light, and even more so the version of it due to Peres and Mermin [1, 5]. The following conclusion seems therefore to be unavoidable: we have to abandon the naive notion of an external reality that exists independently of us. And yet, this conclusion might be nonetheless avoidable. Indeed, let us notice that in order to turn the aforementioned “precept of the founders” into a logical consequence of the quantum formalism, the BKS theorem had to invoke another “precept of the founders”. This precept states that some observables have infinitely sharp eigenvalues. Any deviations from these sharply defined (eigen)values should be attributable to measurement disturbances, i.e., to imperfections of our measuring devices. Alas, the two precepts seem to be in conflict with one another. For, first, we are told to accept that the values of an observable are brought into being by the very act of measurement. Thereafter, we are asked to accept that the values we have recorded by measurement need not always be the “true” ones. In most cases, so we are told, measurements show values that only approximate the “true” ones. Why should we accept this statement without having any compelling evidence of its truth? Paraphrasing Mermin [6], we may perhaps say

that it has been merely reverence for the Patriarchs what diverted people from objecting a precept that appears to be nothing but a misapplication of the other, already accepted one. The word *quantum* reminds us of the strong appeal that sharp, integer values had during the foundational period of QM. The impressive successes of the quantum formalism surely helped to firmly establish the belief on sharp eigenvalues as a mandatory prescription of the quantum creed. The positivist commitment of the founders, which led them to deny the very existence of what has not been measured, was curiously betrayed by the founders themselves, who took for granted the existence of discrete, infinitely sharp, ideal eigenvalues. If we instead consistently rely on measurement outcomes alone, then we have no reason to assume that observables must have infinitely sharp eigenvalues.

Before we analyze the consequences of entertaining the rather unusual assumption of unsharp eigenvalues, let us consider an archetypical measurement, namely that of a spin-1/2. Fig. (1) shows schematically the detector part of a Stern-Gerlach array (SGA). Particles in the spin-up state $|\uparrow\rangle$ produce a click in the (+)-detector, and correspondingly for the state $|\downarrow\rangle$ and the (-)-detector. Submitted to the action of the SGA, a spin-state $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$ is brought into a spin-path entangled state: $|\psi\rangle \rightarrow |\Psi\rangle = a|\uparrow\rangle|+\rangle + b|\downarrow\rangle|-\rangle$, so that the probability that the (+)-detector fires is $p_+ = |a|^2$. Here, it is assumed that the particle beam is well collimated, so that spin-up particles can reach only the (+)-detector. Otherwise, the measurement is unsharp. The SGA can be taken as representative of all measurements. It is by reading some pointer that we fix the value of whatever observable we want to measure. Any pointer has a finite resolution, as it is illustrated in fig. (1) by the lengths Δx_{\pm} . All particles being detected within Δx_+ are assigned the infinitely sharp spin-value +1 (in units of $\hbar/2$). Most particles fall around the middle of the Δx_+ zone. The spatial spreading of the detected particles is attributed to imperfections of the SGA, which includes source and detector parts. Fig. (1) shows two fitted histograms. Let us assume for a moment that these histograms correspond to macroscopic objects, apples of two varieties, for example, grown in two different countries. Instead of having recorded particles' positions x we assume having recorded the weights w of apples in a sample that contains the two varieties. Let w_{\pm} be the two mean values of these weights. If we weigh an apple of the sample and obtain, e.g., $w > w_+$, we do not interpret this outcome by saying that the true value is not w but w_+ , and that any deviation from w_+ must be attributed to an imperfect measurement. This is so because we can weigh the same apple many times, thereby obtaining values such as $w \pm \delta w$, with $\delta w \ll w$, that average out tightly at w . Imagine now that our measuring procedure is such that in order to weigh an apple we must destroy it. In such a case, nothing would prevent

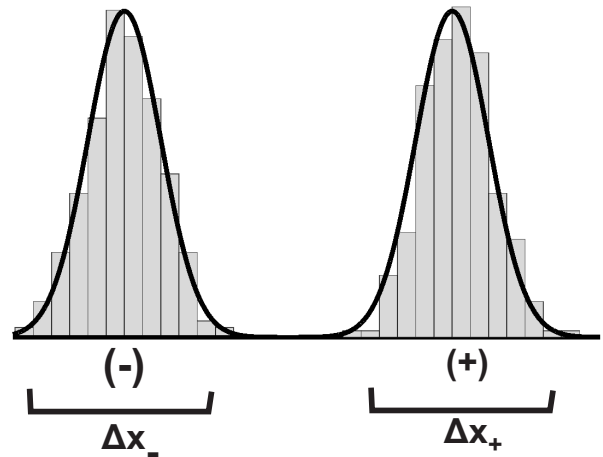


FIG. 1: Schematic representation of the detection part (detector screen) in a Stern-Gerlach array. All spin-1/2 particles falling within the spot labeled Δx_+ (Δx_-) are assigned to the spin-up (spin-down) state. Under the assumption of infinitely sharp spin values, spreads such as Δx_{\pm} are attributed to measurement uncertainties. Unsharp spin values could also explain the observed results, similarly to cases in which one measures classical observables – e.g. weights – of a population containing some two varieties.

us from saying that the spread in weights comes from imperfect measurements and that all the apples in our sample are produced by nature with a weight that is either w_+ or w_- . This is what happened in QM, which originally dealt with microscopic objects that got destroyed when submitted to measurement. Measurement's accuracy was assessed by repeating the experiment on "identically prepared" replicas, thereby taking for granted that, say, electrons are characterized by sharply defined values. Quantum non-demolition measurements were not available at that time. This kind of measurement is now often applied [7], although not with the aim of testing the assumption of infinitely sharp eigenvalues. Anyhow, it is clear that such an assumption is not the only possible one. It might occur that, like apples, also particles possess spin values that could slightly differ from their mean values $\pm\hbar/2$. In the following, we will entertain the assumption that observables have unsharp eigenvalues and study the consequences of this assumption for some tests of quantum contextuality. As we shall see, these consequences can be limited to HV models, leaving QM untouched. The latter remains being what it always has been: an ideal, extremely accurate model of physical reality. In this model, observables are represented by operators whose sharply defined eigenvalues coincide with the mean values of measured observables.

THE MERMIN-PERES VERSION OF THE BELL-KOCHEN-SPECKER THEOREM

Let us address now the Mermin-Peres version of the BKS theorem. It will be convenient to use Mermin's first version of it [1], that we reproduce here for completeness' sake and future reference. This version applies to a four-dimensional Hilbert space that corresponds to two qubits. We write, e.g., X_1 for $\sigma_x^{(1)}$, the Pauli x -matrix of the first qubit. A HV theory ascribes the value $v(O)$ to the observable O . If a set of mutually commuting observables identically satisfy a functional relationship $f(A, B, C, \dots) = 0$, then this relationship must also be satisfied by the assigned values: $f(v(A), v(B), v(C), \dots) = 0$. Thus, it must hold $v(AB) = v(A)v(B)$, whenever $[A, B] = 0$. By considering operator identities such as $(X_1 Y_2)(Y_1 X_2)(Z_1 Z_2) = I$, etc., one gets the Mermin system of equations:

$$v(X_1)v(X_2)v(X_1 X_2) = 1, \quad (1a)$$

$$v(Y_1)v(Y_2)v(Y_1 Y_2) = 1, \quad (1b)$$

$$v(X_1)v(Y_2)v(X_1 Y_2) = 1, \quad (1c)$$

$$v(Y_1)v(X_2)v(Y_1 X_2) = 1, \quad (1d)$$

$$v(X_1 Y_2)v(Y_1 X_2)v(Z_1 Z_2) = 1, \quad (1e)$$

$$v(X_1 X_2)v(Y_1 Y_2)v(Z_1 Z_2) = -1. \quad (1f)$$

The above six equations cannot hold simultaneously. This claim is derived as follows [1]: The assigned values $v(O)$ are such that $v(O) \in \{-1, 1\}$. This is so because in a HV-theory $v(O)$ must be one of the possible measurement outcomes for O . According to QM – and, allegedly, to experimental evidence – these outcomes are O 's eigenvalues. Now, each value appears exactly twice on the left of the above equations. Hence, the product of all values on the left gives 1. Since the product of the right sides is -1 , we get a contradiction. The assignment of values under the above restrictions is thus impossible. Clearly, the restriction $v(O) \in \{-1, 1\}$ plays a key role. To substantiate it, experimental evidence is often invoked. However, what experimental evidence imposes is that $v(O) = \pm 1 + \delta_{\pm}$, for some δ_{\pm} . Let us thus see the consequences of imposing this last restriction instead of $v(O) = \pm 1$.

A MODIFIED MERMIN SET OF EQUATIONS

Of course, besides $v(O) = \pm 1 + \delta_{\pm}$, we must include some additional restrictions, e.g., that X_1 and Y_1 cannot be measured simultaneously. Moreover, all the above values $v(O)$ must have an operational meaning. Taking for example eq. (1a), we assume that it corresponds to an experimental array that is well suited for the measurement of X_1 and X_2 . The value assigned to $X_1 X_2$ is then given by $v(X_1 X_2) = v(X_1)v(X_2)$. Thus, we can

consistently write $v(X_1)v(X_2)v(X_1 X_2) = v(X_1)^2 v(X_2)^2$. If we set $v(X_1) = \pm 1 + \delta_{x1}$ and $v(X_2) = \pm 1 + \delta_{x2}$, then $v(X_1 X_2) = \pm 1 + \delta_{x1x2}$. The value of δ_{x1x2} follows from δ_{x1} and δ_{x2} in a way that the HV-model should prescribe. We consider models for which the assignments $v(X_i) = \pm 1 + \delta_{xi}$ reflect that spin values are unsharp, i.e., spread around the mean values ± 1 , very much like the weights of two apples' varieties. We thus set $\delta_{x1x2} = v_1 \delta_{x2} + \delta_{x1} v_2$ and replace, e.g., eq. (1a) by $(v_1 + \delta_{x1})^2 (v_2 + \delta_{x2})^2 = 1 + 2v_1 v_2 \delta_{x1x2}$, with $v_i = \pm 1$, $i = 1, 2$. In other words, we treat the δ_{xi} as deviations from the corresponding mean values and apply for quantities like $v(X_1)v(X_2)$ the rules of error propagation. Proceeding in this way, instead of eqs. (1) we get the following set of equations:

$$(v_1 + \delta_1)^2 (v_2 + \delta_2)^2 = 1 + 2v_1 v_2 \delta_{12}, \quad (2a)$$

$$(v_3 + \delta_3)^2 (v_4 + \delta_4)^2 = 1 + 2v_3 v_4 \delta_{34}, \quad (2b)$$

$$(v_1 + \delta_1)^2 (v_4 + \delta_4)^2 = 1 + 2v_1 v_4 \delta_{14}, \quad (2c)$$

$$(v_3 + \delta_3)^2 (v_2 + \delta_2)^2 = 1 + 2v_3 v_2 \delta_{32}, \quad (2d)$$

$$(w_1 + \Delta_1)^2 (w_2 + \Delta_2)^2 = 1 + 2w_1 w_2 \Delta_{12}, \quad (2e)$$

$$-(w_1 + \Delta_1)^2 (w_3 + \Delta_3)^2 = -1 + 2w_1 w_3 \Delta_{13}. \quad (2f)$$

Eqs. (2a – 2d) involve the parameters v_j and δ_j , with $j = 1, \dots, 4$, whereas eqs. (2e, 2f) involve the parameters w_j and Δ_j , with $j = 1, 2, 3$. They are defined as follows: $v(X_1) = v_1 + \delta_1$, $v(X_2) = v_2 + \delta_2$, $v(Y_1) = v_3 + \delta_3$, $v(Y_2) = v_4 + \delta_4$; $v(Z_1 Z_2) = w_1 + \Delta_1$, $v(X_1 Y_2) = w_2 + \Delta_2$, $v(Y_1 X_2) = w_3 + \Delta_3$. Here, v_j and w_j take on the values ± 1 , while δ_j and Δ_j are free parameters that besides entering the above equations can be required to satisfy additional constraints, such as $|\delta_j| \leq \epsilon$ and $|\Delta_j| \leq \epsilon$, with $\epsilon \ll 1$. Because $\delta_{jk} = v_j \delta_k + \delta_k v_j$ ($j, k \in \{1, 2, 3, 4\}$) and $\Delta_{jk} = w_j \Delta_k + \Delta_k w_j$ ($j, k \in \{1, 2, 3\}$) we have more free parameters than equations. In fact, for all possible choices of v_j and w_j we can solve eqs. (2a – 2d) by expressing three of the δ_j in terms of the fourth, and solve eqs. (2e, 2f) by expressing two of the Δ_j in terms of the third. In other words, we can always obtain values for the δ_j and Δ_j so that they satisfy the above equations, alongside with $|\delta_j| \leq \epsilon$ and $|\Delta_j| \leq \epsilon$. As an example, we set $v_1 = -1$, $v_2 = 1$, $v_3 = 1$, $v_4 = -1$, $\epsilon = 10^{-3}$ and obtain, among other choices, $\delta_1 = 0.887444 \times 10^{-4}$, $\delta_2 = 0.23779 \times 10^{-4}$, $\delta_3 = -0.63717 \times 10^{-7}$, $\delta_4 = -0.23779 \times 10^{-4}$, while setting $w_1 = 1$, $w_2 = -1$, $w_3 = -1$, we obtain $\Delta_1 = -0.15470 \times 10^{-3}$, $\Delta_2 = -0.57722 \times 10^{-3}$ and $\Delta_3 = 0.15469 \times 10^{-3}$. We have thus exhibited a consistent assignment of values for the set of observables entering the Peres-Mermin version of the KBS theorem.

Let us stress that eqs. (2) follow from very general assumptions. Indeed, while we have set $v(AB) = v(A)v(B)$ for some commuting observables A, B , we have not assumed that such a product rule holds for non-commuting observables. Had we done so, then we would have run

into contradictions. Indeed, from an operator identity such as $[\sigma_x, \sigma_y] = 2i\sigma_z$, it would follow that $v(\sigma_x)v(\sigma_y) - v(\sigma_y)v(\sigma_x) = 0 = 2iv(\sigma_z)$, which cannot hold together with $v(\sigma_z) = \pm 1$. For this reason, we cannot consistently apply the product rule for all the equations in the Mermin system, eqs. (1). For example, we cannot set $v(X_1X_2) = v(X_1)v(X_2)$ in eq. (1a) and simultaneously $v(Y_1Y_2) = v(Y_1)v(Y_2)$ in eq. (1b). This is also not required when proving the Peres-Mermin theorem. Note that while we refrain from applying the product rule, this does not make the model contextual. We do not apply the product rule because otherwise the model would be inconsistent. For the very same reason we do not set for $v(X_1X_2)$ in eq. (1f) a value that derives from the values $v(X_1) = v_1 + \delta_1$ and $v(X_2) = v_2 + \delta_2$ entering eq. (1a) (cf. eq. (2a)). Eqs. (1e,1f) are thus set apart from eqs. (1a – 1d), in the sense that they are related to quite different and independent experimental arrays. Indeed, let us consider the observables X_1Y_2 , Y_1X_2 and Z_1Z_2 entering eq. (1f). Because any one of them is the product of the other two, we need to measure only two of them and then apply the product rule. These two observables constitute a complete set of commuting observables, i.e., by fixing their eigenvalues we fix the corresponding common eigenvector. Written in terms of the eigenvectors of Pauli- Z , i.e., $Z|\pm\rangle = \pm|\pm\rangle$, the eigenvectors of the above observables read $|\Phi^\pm\rangle = (\pm i|++\rangle + |--\rangle)/\sqrt{2}$, $|\Psi^\pm\rangle = (\pm i|+-\rangle + |-+\rangle)/\sqrt{2}$. That is, they constitute a Bell-like basis. If we want to measure, say, Z_1Z_2 and X_1Y_2 , we must set up an array that performs projective measurements represented by the four projectors $|\Phi^\pm\rangle\langle\Phi^\pm|$ and $|\Psi^\pm\rangle\langle\Psi^\pm|$. If, for instance, the detector $|\Phi^+\rangle\langle\Phi^+|$ fires, we make the assignments $v(Z_1Z_2) = +1$ and $v(X_1Y_2) = -1$, while we assign to the third observable, Y_1X_2 , a value that equals the product of the measured ones: $v(Y_1X_2) = v(Z_1Z_2)v(X_1Y_2) = -1$, and so on. All this holds under the assumption of infinitely sharp eigenvalues. Assuming instead unsharp eigenvalues, we set $v(Z_1Z_2) = 1 + \Delta_1$. Eqs. (2) refer to this case. In particular, eq. (2f) comes from considering the identity $(Z_1Z_2)(Y_1Y_2) = -X_1X_2$ and from assuming that our measuring device projects onto the common eigenvectors of Z_1Z_2 and Y_1Y_2 .

He have thus derived eqs. (2) by assuming realism and non-contextuality, besides unsharp eigenvalues. We introduced as much free parameters ($\delta_{i=1,\dots,4}$, $\Delta_{j=1,2,3}$) as these assumptions allow. One could wonder if further constraints on these parameters could arise from the uncertainty relations. We know that if we measure two non-commuting observables such as σ_x and σ_y on identically prepared systems, the respective outcomes fulfill uncertainty relations. The general form of these relations reads $(\Delta A)^2(\Delta B)^2 \geq [\langle C \rangle^2 + \langle F \rangle^2]/4$, with $[A, B] = iC$, $F = AB + BA - 2\langle A \rangle \langle B \rangle$ and $(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2$. Setting $A = \sigma_x$, $B = \sigma_y$ we get $\langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 \leq 1$, a

condition that is clearly satisfied no matter which state, pure or mixed, is submitted to measurement. In any case, this condition imposes no further restrictions on the values of δ_1 and δ_3 , which correspond to X_1 and Y_1 , respectively. Similar considerations can be made for the other parameters.

In the special case when non-contextuality stems from locality – i.e., if measurements are performed at space-like separated locations – we can derive restricted forms of the BKS-theorem that are expressed in terms of inequalities [1]. A well known one is the Clauser-Horne-Shimony-Holt (CHSH) inequality [8]. In contrast to the BKS-theorem, which holds for arbitrary states, the CHSH-inequality holds for maximally entangled states and involves four observables, A_1 , A'_1 , B_2 and B'_2 , whose eigenvalues (a_1 , a'_1 , b_2 , b'_2) are ± 1 . Last restriction implies that $(a_1 + a'_1)b_2 + (a_1 - a'_1)b'_2 = \pm 2$. From this, one readily derives the CHSH-inequality $|\langle A_1B_2 \rangle + \langle A'_1B_2 \rangle + \langle A_1B'_2 \rangle - \langle A'_1B'_2 \rangle| \leq 2$, which QM violates for appropriate choices of the involved states and observables. The assumption of infinitely sharp eigenvalues plays an essential role here as well. By dropping it we should be able to explain any experimental outcomes, as we have enough free parameters at our disposal. Similar considerations should apply to other versions of Bell-like inequalities and to different variants of the BKS theorem [9–11].

CONCLUSIONS

As we have seen, the assumption of unsharp eigenvalues has far-reaching consequences for some fundamental tests of QM. A related but quite different subject is that of unsharp measurements. As an example of the latter we may refer to disturbances that could restrict the compatibility of observables being submitted to sequential measurements [4, 7]. One can take these disturbances into account and still produce results that HV models cannot explain [7]. Our approach differs also from earlier ones that addressed finite precision measurements [12, 13]. It has been shown that finite precision does not nullify the BKS theorem, but rather hints at a different type of contextuality, called “existential contextuality” [14]. The consequences of assuming unsharp eigenvalues seem to have been neglected. While finite precision measurements might spoil our ability to meet the benchmark set by fundamental tests of QM, the assumption of unsharp eigenvalues spoils the benchmark itself. Unsharp eigenvalues surely fit among the assumptions of HV theories and, moreover, they are not alien to QM. Indeed, let us recall some representative cases: Atomic energy spectra have discrete as well as continuous – i.e., unsharp – parts, whereas in more complex systems such as semiconductors one often deals with energy bands. Faced with the natural linewidth of spontaneously emitted light, one realizes

that atomic energy states in the discrete part of the spectrum cannot be infinitely sharp. The spread ΔE of a level can be traced back to the coupling between atomic electrons and electromagnetic fields that have continuous energy spectra. The coupling can then modify an otherwise discrete part of the spectrum. Moreover, this coupling involves electron's charge as much as its magnetic moment μ . Elementary particles are not characterized by a fixed charge's value, as it was originally assumed. Since long, "running coupling constants" are routinely employed in high-energy physics. Hence, it is not physically unreasonable to assume a spread $\Delta\mu$, which in turn implies a spread of spin's eigenvalues. But independently of any plausibility arguments, the fact is that there is a spread of recorded values, which may be attributed to the quantity being measured rather than to imperfections of the measuring procedure. A consistent realist theory may be built upon such an assumption. The BKS ban does not apply under such circumstances, and values can be assigned to observables without running into contradictions. We could however hope to rule out HV theories by addressing observables whose eigenvalues span a continuous range. In fact, CHSH-like inequalities have been derived for such a case, as for instance in [15], where eigenvalues are given by $\cos\theta$, with $\theta \in [0, \pi]$. Now, inequalities follow from the fact that such eigenvalues are bounded: $|\cos\theta| \leq 1$. Thus, the role that was previously played by infinitely sharp eigenvalues, is now played by an infinitely sharp boundary. Clearly, a HV model could here again be constructed upon the assumption that the boundary is unsharp, an assumption that would be surely in agreement with experimental facts.

Finally, let us notice that foundational issues such as those discussed in this work might be relevant for quantum information theory as well [16, 17]. Any quantum

device works with inherent uncertainties of the kind illustrated by the SGA we have considered here. Hence, classical analogs that mimic unsharp quantum eigenvalues could shed light on several issues of current interest [18]. Of particular relevance in this respect is the recent identification of quantum contextuality as a critical resource for quantum speed-up of fault-tolerant quantum computation [19].

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- [1] N. D. Mermin, Phys. Rev. Lett. 65 (1990) 3373.
 - [2] J. S. Bell, Rev. Mod. Phys. 38 (1966) 447.
 - [3] S. Kochen and E. P. Specker, J. Math. Mech. 17 (1967) 59.
 - [4] J. Szangolies, M. Kleinmann, and O. Gühne, Phys. Rev. A 87 (2013) 050101(R).
 - [5] A. Peres, J. Phys. A 24 (1991) L175.
 - [6] N. D. Mermin, Rev. Mod. Phys. 65 (1993) 803.
 - [7] G. Kirchmair *et al.*, Nature (London) 460 (2009) 494.
 - [8] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23 (1969) 880.
 - [9] S. Yu and C. H. Oh, Phys. Rev. Lett. 108 (2012) 030402.
 - [10] C. Zu *et al.*, Phys. Rev. Lett. 109 (2012) 150401.
 - [11] G. Cañas *et al.*, Phys. Rev. A 90 (2014) 012119.
 - [12] D. A. Meyer, Phys. Rev. Lett. 83 (1999) 3751.
 - [13] A. Kent, Phys. Rev. Lett. 83 (1999) 3755.
 - [14] D. M. Appleby, Phys. Rev. A 65 (2002) 022105.
 - [15] C. V. S. Borges, P. Milman, and A. Keller, Phys. Rev. A 86 (2012) 052107.
 - [16] K. Nagata, Phys. Rev. A 52 (2005) 012325.
 - [17] N. Aharon and L. Vaidman, Phys. Rev. A 77 (2008) 052310.
 - [18] H. Buhrman *et al.*, Rev. Mod. Phys. 82 (2010) 665.
 - [19] M. Howard *et al.*, Nature (London) 510 (2014) 351.